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A NOTE ON DEGENERATE STIRLING NUMBERS OF THE FIRST KIND

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ABSTRACT. Recently, the degenerate Stirling numbers of the first kind were introduced. In this paper, we give some formulas for the degenerate Stirling numbers of the first kind in the terms of the complete Bell polynomials with higher-order harmonic number arguments. Also, we derive an identity connecting the degenerate Stirling numbers of the first kind and the degenerate derangement numbers by using probabilistic method.

1. Introduction

The numbers $S_1(n,k)$ and $S_2(n,k)$ are, in the notation of Riordan [13,14], Stirling numbers of the first kind and of the second kind, respectively and they are given by:

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k, \ (n \geq 0), \tag{1.1}$$

$$x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \ (n \geq 0), \quad (\text{see } [1,2,3]). \tag{1.2}$$

where $(x)_n = x(x-1)\cdots(x-(n-1))$, $(n \geq 1)$, and $(x)_0 = 1$.

The unsigned Stirling numbers of the first kind are defined as

$$< x >_n = \sum_{k=0}^{n} S(n,k)x^k, \ (n \geq 0), \quad (\text{see } [13]), \tag{1.3}$$

where $< x >_0 = 1$, $< x >_n = x(x+1)\cdots(x+n-1), (n \geq 1)$.

From (1.1) and (1.3), we note that

$$S(n,k) = (-1)^{n-k} S_1(n,k), \ (n \geq k \geq 0). \tag{1.4}$$

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Let λ be a real number. Then the degenerate Euler polynomials are defined by the generating function

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!},\quad (\text{see [1]}). \tag{1.5}$$

When $x=0$, $\mathcal{E}_{n,\lambda}=\mathcal{E}_{n,\lambda}(0)$ are called the degenerate Euler numbers.

From (1.5), we note that $\lim_{\lambda\to 0}\mathcal{E}_{n,\lambda}(x)=E_n(x)$, $(n\geq 0)$, where $E_n(x)$ are the Euler polynomials given by

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\quad (\text{see [2, 12, 13, 14]}).$$

By (1.1) and (1.3), we easily get

$$\frac{1}{k!}\Big(\log(1+t)\Big)^k=\sum_{n=k}^{\infty}S_1(n,k)\frac{t^n}{n!},\quad (\text{see [13]}), \tag{1.6}$$

and

$$\frac{1}{k!}\Big(\log\big(\frac{1}{1-t}\big)\Big)^k=\sum_{n=k}^{\infty}S(n,k)\frac{t^n}{n!}. \tag{1.7}$$

For $\lambda\in\mathbb{R}$, the degenerate falling factorial sequence is defined as

$$(x)_{0,\lambda}=1,\ (x)_{n,\lambda}=x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda),\ (n\geq 1). \tag{1.8}$$

Note that $\lim_{\lambda\to 1}(x)_{n,\lambda}=(x)_n$, $\lim_{\lambda\to 0}(x)_{n,\lambda}=x^n$. In [5], the degenerate Stirling numbers of the first kind were defined by Kim as

$$(x)_{n,\lambda}=\sum_{k=0}^{n}S_{1,\lambda}(n,k)x^k,\ (n\geq 0). \tag{1.9}$$

Kölbig gave the following formula:

$$\frac{d^r}{dx^r}e^{f(x)}=e^{f(x)}Bel_r\left(f^{(1)}(x),f^{(2)}(x),\cdots,f^{(r)}(x)\right),\quad (\text{see [10, 11]}), \tag{1.10}$$

where $f^{(r)}(x)=\left(\frac{d}{dr}\right)^r f(x)$ and $Bel_r(x_1,x_2,\cdots,x_r)$ are the complete Bell polynomials given by

$$Bel_r(x_1,x_2,\cdots,x_r)=\sum_{k_1+2k_2+\cdots+rk_r=r}\binom{r}{k_1,\cdots,k_r}\left(\frac{x_1}{1!}\right)^{k_1}\left(\frac{x_2}{2!}\right)^{k_2}\cdots\left(\frac{x_r}{r!}\right)^{k_r}.$$

The complete Bell polynomials are also given by the exponential generating function

$$\exp\left(\sum_{j=1}^{\infty}x_j\frac{t^j}{j!}\right)=\sum_{n=0}^{\infty}Bel_n(x_1,x_2,\cdots,x_n)\frac{t^n}{n!},\quad (\text{see [4]}). \tag{1.11}$$

In [4], the complete degenerate Bell polynomials are defined by

$$\exp\left(\sum_{j=1}^{\infty} x_j (1)_{j,\lambda} \frac{t^j}{j!}\right) = \sum_{n=0}^{\infty} Bel_n^{(\lambda)}(x_1, x_2, \cdots, x_n)\frac{t^n}{n!}, \tag{1.12}$$

where

$$Bel_n^{(\lambda)}(x_1, \cdots, x_n) = \sum_{k_1+2k_2\cdots+nk_n=n} \binom{n}{k_1, \cdots, k_n} \left(\frac{x_1(1)_{1,\lambda}}{1!}\right)^{k_1} \cdots \left(\frac{x_n(1)_{n,\lambda}}{n!}\right)^{k_n}.$$

From (1.12), we note that

$$Bel_n^{(\lambda)}(1, 1, \cdots, 1) = Bel_{n,\lambda}$$

are the degenerate Bell numbers given by

$$e^{(1+\lambda t)^{\frac{1}{\lambda}}-1} = \sum_{n=0}^{\infty} Bel_{n,\lambda}\frac{t^n}{n!}, \quad \text{(see [9])}. \tag{1.13}$$

Here the finite Hurwitz-type functions of order r are defined as

$$\sum_{k=0}^{n-1} \frac{1}{(k+x)^r} = H_n^{(r)}(x), \ (n, r \in \mathbb{N}). \tag{1.14}$$

When $x = 1$, $H_n^{(r)} = H_n^{(r)}(1) = \sum_{k=0}^{n-1} \frac{1}{(k+1)^r} = \sum_{k=1}^{n} \frac{1}{k^r}$ are the harmonic numbers of order r.

From (1.9), we can derive the unsigned Stirling numbers of the first kind as follows:

$$< x >_{n,\lambda} = \sum_{k=0}^{n} S_\lambda(n, k)x^k, \quad \text{(see [5, 6])}, \tag{1.15}$$

where $< x >_{0,\lambda} = 1$, $< x >_{n,\lambda} = x(x+\lambda)\cdots(x+(n-1)\lambda), (n \geq 1)$. By (1.9) and (1.15), we get

$$S_\lambda(n, k) = (-1)^{n-k} S_{1,\lambda}(n, k), \ (n \geq k \geq 0). \tag{1.16}$$

In this paper, we give some formulas for the degenerate Stirling numbers of the first kind in the terms of the complete Bell polynomials with higher-order harmonic number arguments. Also, we derive an identity connecting the degenerate Stirling numbers of the first kind and the degenerate derangement numbers by using probabilistic method.

2. Degenerate Stirling numbers of the first kind

From (1.1), we have

$$\begin{aligned}(1+\lambda t)^{\frac{x}{\lambda}} &= \sum_{k=0}^{\infty}(x)_{k,\lambda}\frac{t^k}{k!} = \sum_{k=0}^{\infty}\sum_{n=0}^{k} S_{1,\lambda}(k,n)x^n\frac{t^k}{k!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=n}^{\infty} S_{1,\lambda}(k,n)\frac{t^k}{k!}\right)x^n.\end{aligned} \tag{2.1}$$

On the other hand

$$\begin{aligned}(1+\lambda t)^{\frac{x}{\lambda}} &= \sum_{n=0}^{\infty}\left(\frac{x}{\lambda}\right)^n \frac{1}{n!}\Big(\log(1+\lambda t)\Big)^n \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\Big(\log(1+\lambda t)^{\frac{1}{\lambda}}\Big)^n x^n.\end{aligned} \tag{2.2}$$

By (2.1) and (2.2), we get

$$\frac{1}{n!}\Big(\log(1+\lambda t)^{\frac{1}{\lambda}}\Big)^n = \sum_{k=n}^{\infty} S_{1,\lambda}(k,n)\frac{t^k}{k!}. \tag{2.3}$$

It is easy to show that

$$(-x)_{n,\lambda} = (-1)^n < x >_{n,\lambda},\ (n \geq 1). \tag{2.4}$$

Thus, by (2.4), we see again that

$$S_{\lambda}(n,k) = (-1)^{n-k} S_{1,\lambda}(n,k),\ (n \geq k \geq 0). \tag{2.5}$$

From (2.4), we also note that

$$\frac{1}{k!}\left(\log\Big(\frac{1}{(1-\lambda t)^{\frac{1}{\lambda}}}\Big)\right)^k = \sum_{n=k}^{\infty} S_{\lambda}(n,k)\frac{t^n}{n!}. \tag{2.6}$$

By (1.5), we easily get

$$2\sum_{l=0}^{m-1}(1+\lambda t)^{\frac{l}{\lambda}}(-1)^l = \sum_{n=0}^{\infty}\Big(\mathcal{E}_{n,\lambda}+\mathcal{E}_{n,\lambda}(m)\Big)\frac{t^n}{n!}, \tag{2.7}$$

where $m \in \mathbb{N}$ with $m \equiv 1 \pmod 2$.

On the other hand

$$\begin{aligned}2\sum_{l=0}^{m-1}(1+\lambda t)^{\frac{l}{\lambda}}(-1)^l &= \sum_{n=0}^{\infty}\left(2\sum_{l=0}^{m-1}(l)_{n,\lambda}(-1)^l\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(2\sum_{k=0}^{n}\sum_{l=0}^{m-1}(-1)^l S_{1,\lambda}(n,k)l^k\right)\frac{t^n}{n!}.\end{aligned} \tag{2.8}$$

Thus, by (2.7) and (2.8), we get

$$\mathcal{E}_{n,\lambda} + \mathcal{E}_{n,\lambda}(m) = 2\sum_{k=0}^{n}\sum_{l=0}^{m-1}(-1)^l S_{1,\lambda}(n,k)l^k, \tag{2.9}$$

where $n \geq 0$, and $m \in \mathbb{N}$, with $m \equiv 1 \pmod 2$.

We summarize this as a theorem.

Theorem 2.1. *Let $S_{1,\lambda}(n,k)$ be the degenerate Stirling numbers of the first kind given in* (1.9) *or* (2.3)*. Then we have*

$$\mathcal{E}_{n,\lambda} + \mathcal{E}_{n,\lambda}(m) = 2\sum_{k=0}^{n}\sum_{l=0}^{m-1}(-1)^l S_{1,\lambda}(n,k)l^k,$$

where $n \geq 0$, and $m \in \mathbb{N}$, with $m \equiv 1$ (mod 2).

Now, we observe that

$$\begin{aligned}\frac{d}{dx}\frac{<x>_{n,\lambda}}{x} &= \frac{d}{dx}e^{\left(\log(<x>_{n,\lambda} - \log x\right)} \\ &= \frac{<x>_{n,\lambda}}{x}\frac{d}{dx}\left(\sum_{k=0}^{n-1}\log(x+k\lambda) - \log x\right) \\ &= \frac{<x>_{n,\lambda}}{x}\sum_{k=1}^{n-1}\frac{1}{x+k\lambda}.\end{aligned} \tag{2.10}$$

Let $T_\lambda(x) = \sum_{k=1}^{n-1}\frac{1}{x+k\lambda}$. Then we have

$$\frac{d^r}{dx^r}T_\lambda(x) = T_\lambda^{(r)}(x) = (-1)^r r!\sum_{k=1}^{n-1}\frac{1}{(x+k\lambda)^{r+1}}. \tag{2.11}$$

By (2.11), we get

$$T_\lambda^{(r)}(0) = (-1)^r r!\lambda^{-r-1}\sum_{k=1}^{n-1}\frac{1}{k^{r+1}} = (-1)^r r!\lambda^{-r-1}H_{n-1}^{(r+1)}. \tag{2.12}$$

From (1.10), we have

$$\frac{d^r}{dx^r}\frac{<x>_{n,\lambda}}{x} = \frac{<x>_{n,\lambda}}{x}Bel_r\left(T_\lambda(x), T_\lambda^{(1)}(x), \cdots, T_\lambda^{(r-1)}(x)\right). \tag{2.13}$$

By (2.13), we get

$$\begin{aligned}&\frac{d^r}{dx^r}\frac{<x>_{n,\lambda}}{x}\bigg|_{x=0}\\&=(n-1)!\lambda^{n-1}Bel_r\left(\frac{1}{\lambda}H_{n-1}^{(1)},\frac{(-1)1!}{\lambda^2}H_{n-1}^{(2)},\cdots,\frac{(-1)^{r-1}(r-1)!}{\lambda^r}H_{n-1}^{(r)}\right).\end{aligned}\tag{2.14}$$

The equation (2.14) is equivalent to

$$\begin{aligned}&\frac{d^r}{dx^r}\frac{<x>_{n,\lambda}}{x}\bigg|_{x=0}\\&=(n-1)!\lambda^{n-1-r}Bel_r\left(H_{n-1}^{(1)},(-1)1!H_{n-1}^{(2)},\cdots,(-1)^{r-1}(r-1)!H_{n-1}^{(r)}\right).\end{aligned}\tag{2.15}$$

On the other hand,

$$\begin{aligned}\frac{d^r}{dx^r}\frac{<x>_{n,\lambda}}{x}&=\frac{d^r}{dx^r}\sum_{l=0}^{n}S_\lambda(n,l)x^{l-1}\\&=\sum_{l=0}^{n}S_\lambda(n,l)(l-1)(l-2)\cdots(l-r)x^{l-1-r}.\end{aligned}\tag{2.16}$$

Thus, we have

$$\frac{d^r}{dx^r}\frac{<x>_{n,\lambda}}{x}\bigg|_{x=0}=S_\lambda(n,r+1)r!,\tag{2.17}$$

where $n\geq r+1\geq 1$.

Therefore, by (2.15) and (2.17), we obtain the following equation.

$$\begin{aligned}&S_\lambda(n,r+1)\\&=\frac{(n-1)!}{r!}\lambda^{n-1-r}Bel_r\left(H_{n-1}^{(1)},(-1)1!H_{n-1}^{(2)},\cdots,(-1)^{r-1}(r-1)!H_{n-1}^{(r)}\right).\end{aligned}\tag{2.18}$$

In particular, by replacing n by $n+1$, we get

$$S_\lambda(n+1,r+1)=\frac{n!}{r!}\lambda^{n-r}Bel_r\left(H_n^{(1)},(-1)1!H_n^{(2)},\cdots(-1)^{r-1}(r-1)!H_n^{(r)}\right),\tag{2.19}$$

where $n\geq r\geq 0$.

Thus we obtain the following theorem.

Theorem 2.2. *Let $S_\lambda(n,r)$ be the unsigned degenerate Stirling numbers of the first kind given in* (1.15) *or* (2.6). *Then, for $n \geq r \geq 0$, we have*

$$S_\lambda(n+1,r+1) = \frac{n!}{r!}\lambda^{n-r}Bel_r\left(H_n^{(1)},(-1)1!H_n^{(2)},\cdots(-1)^{r-1}(r-1)!H_n^{(r)}\right),$$

where $H_n^{(i)} = \sum_{k=1}^{n}\frac{1}{k^i}$ are the harmonic numbers of order i.

The exponential partial Bell polynomials are the ones given by

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty}x_m\frac{t^m}{m!}\right)^k = \sum_{n=k}^{\infty}B_{n,k}(x_1,x_2,\cdots,x_{n-k+1})\frac{t^n}{n!},\ (k \geq 0), \tag{2.20}$$

where

$$B_{n,k}(x_1,x_2,\cdots,x_{n-k+1}) = \sum_{\substack{i_1+\cdots+i_{n-k+1}=k \\ i_1+2i_2+\cdots+(n-k+1)i_{n-k+1}=n}}\binom{n}{i_1,\cdots,i_{n-k+1}}\left(\frac{x_1}{1!}\right)^{i_1}\cdots\left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{i_{n-k+1}}.$$

Recently, Kim defined the degenerate Stirling numbers of the second kind as follows:

$$\frac{1}{k!}\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)^k = \sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!}, \quad \text{(see [6])}. \tag{2.21}$$

From (2.21), the exponential partial λ-Bell polynomials are considered by Kim as

$$\frac{1}{k!}\left(\sum_{i=1}^{\infty}(1)_{i,\lambda}x_i\frac{t^i}{i!}\right)^k = \sum_{n=k}^{\infty}B_{n,k}^{(\lambda)}(x_1,\cdots,x_{n-k+1})\frac{t^n}{n!}. \tag{2.22}$$

Thus, by (2.22), we get

$$B_{n,k}^{(\lambda)}(x_1,\cdots,x_{n-k+1}) = B_{n,k}((1)_{1,\lambda}x_1,(1)_{2,\lambda}x_2,\cdots,(1)_{n-k+1,\lambda}x_{n-k+1}), \tag{2.23}$$

where $n \geq k \geq 0$.

Note that

$$B_{n,k}(1,1,\cdots,1) = S_2(n,k),$$

and

$$B_{n,k}^{(\lambda)}(1,1,\cdots,1) = S_{2,\lambda}(n,k),\ (n \geq k \geq 0).$$

Kim defined the partially degenerate Bell polynomials which are given by

$$e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)} = \sum_{n=0}^{\infty}Bel_{n,\lambda}(x)\frac{t^n}{n!}, \quad \text{(see [8])}. \tag{2.24}$$

We note that

$$\begin{aligned}\exp\left(\sum_{j=1}^{\infty} x_j(1)_{j,\lambda}\frac{t^j}{j!}\right) &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{j=1}^{\infty} x_j(1)_{j,\lambda}\frac{t^j}{j!}\right)^k \\ &= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} B_{n,k}^{(\lambda)}(x_1, x_2, \cdots x_{n-k+1})\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} B_{n,k}^{(\lambda)}(x_1, x_2, \cdots, x_{n-k+1})\right)\frac{t^n}{n!}.\end{aligned} \tag{2.25}$$

Therefore, by (1.12) and (2.25), we get

$$Bel_n^{(\lambda)}(x_1, x_2, \cdots, x_n) = \sum_{k=0}^{n} B_{n,k}^{(\lambda)}(x_1, x_2, \cdots, x_{n-k+1}). \tag{2.26}$$

Note that

$$\begin{aligned}Bel_n^{(\lambda)}(x, x, \cdots, x) &= \sum_{k=0}^{n} x^k B_{n,k}((1)_{1,\lambda}, (1)_{2,\lambda}, \cdots, (1)_{n-k+1,\lambda}) \\ &= Bel_{n,\lambda}(x), \ (n \geq 0).\end{aligned}$$

Now, we observe that

$$\begin{aligned}&\sum_{n=k}^{\infty} B_{n,k}(0!, 1!\lambda, 2!\lambda^2, \cdots, (n-k)!\lambda^{n-k})\frac{t^n}{n!} \\ &= \frac{1}{k!}\Big(t + \frac{\lambda}{2}t^2 + \frac{\lambda^2}{3}t^3 + \cdots\Big)^k \\ &= \frac{1}{k!}\Big(-\frac{1}{\lambda}\log(1-\lambda t)\Big)^k = \frac{1}{k!}\left(\log\big(\frac{1}{(1-\lambda t)^{\frac{1}{\lambda}}}\big)\right)^k \\ &= \sum_{n=k}^{\infty} S_\lambda(n,k)\frac{t^n}{n!}.\end{aligned} \tag{2.27}$$

Comparing the coefficients on both sides of (2.27), we have

$$B_{n,k}(0!, 1!\lambda, 2!\lambda^2, \cdots, (n-k)!\lambda^{n-k}) = S_\lambda(n,k), \ (n \geq k \geq 0). \tag{2.28}$$

Theorem 2.3. *Let $S_\lambda(n,r)$ be the unsigned degenerate Stirling numbers of the first kind given in* (1.15) *or* (2.6). *Then, for $n \geq k \geq 0$, we have*

$$S_\lambda(n,k) = B_{n,k}(0!, 1!\lambda, 2!\lambda^2, \cdots, (n-k)!\lambda^{n-k}), \tag{2.29}$$

where $B_{n,k}(x_1, x_2, \cdots, x_{n-k+1})$ are the exponential partial Bell polynomials given in (2.20)

3. Further remarks

A derangement is a permutation with no fixed points. For example, (2,3,1) and (3,2,1) are derangements of (1,2,3). But (3,2,1) is not because 2 is a fixed point. The number of derangements of an n-element set is called the n-th derangement number and denoted by d_n, $(n \geq 0)$. The derangement number d_n satisfies the following recurrence relation:

$$d_n = nd_{n-1} + (-1)^n, \ (n \geq 1). \tag{3.1}$$

The generating function of derangement numbers is given by

$$\frac{1}{1-t}e^{-t} = \sum_{n=0}^{\infty} d_n \frac{t^n}{n!}. \tag{3.2}$$

Thus, by (3.2), we easily get

$$d_n = n! \sum_{m=0}^{n} \frac{(-1)^m}{m!}, \ (n \geq 0). \tag{3.3}$$

For $\lambda \in (0,1)$, the degenerate derangement numbers are defined by the generating function

$$\frac{1}{1-\lambda-t}(1+\lambda t)^{-\frac{1}{\lambda}} = \sum_{n=0}^{\infty} d_{n,\lambda} \frac{t^n}{n!}. \tag{3.4}$$

Note that $\lim_{\lambda\to 0} d_{n,\lambda} = d_n$, $(n \geq 0)$. From (3.4), we have

$$\begin{aligned} \sum_{n=0}^{\infty} < 1 >_{n,\lambda} (-1)^n \frac{t^n}{n!} &= (1+\lambda t)^{-\frac{1}{\lambda}} = \left(\sum_{m=0}^{\infty} d_{m,\lambda} \frac{t^m}{m!} \right)(1-\lambda-t) \\ &= (1-\lambda)\sum_{n=0}^{\infty} d_{n,\lambda}\frac{t^n}{n!} - \sum_{n=1}^{\infty} nd_{n-1,\lambda}\frac{t^n}{n!}. \end{aligned} \tag{3.5}$$

By (3.5), we easily get

$$d_{0,\lambda} = \frac{1}{1-\lambda}, \ (1-\lambda)d_{n,\lambda} = (-1)^n < 1 >_{n,\lambda} + nd_{n-1,\lambda}, \ (n \geq 1), \tag{3.6}$$

and

$$(1-\lambda)d_{n+1,\lambda} = (n+\lambda)d_{n,\lambda} + nd_{n-1,\lambda} + n\lambda(-1)^{n-1} < 1 >_{n,\lambda}, \ (n \geq 0). \tag{3.7}$$

For $\lambda \in (0,\infty)$ and $\alpha(>0) \in \mathbb{R}$, the degenerate gamma function is defined as

$$\Gamma_\lambda(\alpha) = \int_0^\infty (1+\lambda t)^{-\frac{1}{\lambda}} t^{\alpha-1} dt, \quad \text{(see [7])}. \tag{3.8}$$

Thus, by (3.8), we get

$$\Gamma_\lambda(\alpha+1) = \frac{\alpha}{(1-\lambda)^{\alpha-1}}\Gamma_{\frac{\lambda}{1-\lambda}}(\alpha), \quad \text{(see [7])}, \tag{3.9}$$

where $\lambda \in (0,1)$ and $0 < \alpha < \frac{1-\lambda}{\lambda}$.

From (3.9), we note that

$$\Gamma_\lambda(k) = \frac{(k-1)!}{(1-\lambda)(1-2\lambda)\cdots(1-k\lambda)}, \ (k \in \mathbb{N},\ \lambda \in (0, \tfrac{1}{k})). \tag{3.10}$$

Let $\lambda \in (0,\infty)$. Then X_λ is the degenerate gamma random variable with parameters $\alpha(>0)$, $\beta(>0)$, if its probability density function f is given by

$$f(x) = \begin{cases} \frac{1}{\Gamma_\lambda(\alpha)}\beta(\beta x)^{\alpha-1}(1+\lambda x)^{-\frac{1}{\lambda}}, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.11}$$

Now, we observe that

$$\begin{aligned} (1+\lambda x)^{\frac{t}{\lambda}} &= \sum_{k=0}^{\infty} \lambda^{-k}\frac{t^k}{k!}\Big(\log(1+\lambda x)\Big)^k \\ &= \sum_{k=0}^{\infty} t^k \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{x^n}{n!}. \end{aligned} \tag{3.12}$$

Assume that $X = X_\lambda$ is the degenerate gamma random variable with parameters 1,1. Then the k-th moment of X is given by

$$\begin{aligned} E[X^k] &= \int_0^\infty x^k \frac{1}{\Gamma_\lambda(1)}(1+\lambda x)^{-\frac{1}{\lambda}}dx \\ &= \frac{1}{\Gamma_\lambda(1)}\Gamma_\lambda(k+1),\ (k \in \mathbb{N}). \end{aligned} \tag{3.13}$$

For the following discussion, we assume that $\lambda \in (0, \frac{1}{k+1})$, and that $t < 1-\lambda$.

On the one hand, the expectation of $(1+\lambda X)^{\frac{t}{\lambda}}$ is given by

$$\begin{aligned} & E[(1+\lambda X)^{\frac{t}{\lambda}}] = \sum_{n=0}^{\infty}\left(\sum_{k=n}^{\infty} S_{1,\lambda}(k,n)\frac{1}{k!}E[X^k]\right)t^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=n}^{\infty} S_{1,\lambda}(k,n)\frac{1}{k!}\left(\frac{\Gamma_\lambda(k+1)}{\Gamma_\lambda(1)}\right)\right)t^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=n}^{\infty} S_{1,\lambda}(k,n)\frac{1}{k!} \times \frac{k!}{\Gamma_\lambda(1)(1-\lambda)(1-2\lambda)\cdots(1-(k+1)\lambda)}\right)t^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=n}^{\infty} \frac{S_{1,\lambda}(k,n)}{(1-2\lambda)(1-3\lambda)\cdots(1-(k+1)\lambda)}\right)t^n. \end{aligned} \tag{3.14}$$

On the other hand,

$$
\begin{aligned}
E[(1+\lambda X)^{\frac{t}{\lambda}}] &= \int_0^\infty (1+\lambda x)^{\frac{t}{\lambda}} \frac{1}{\Gamma_\lambda(1)}(1+\lambda x)^{-\frac{1}{\lambda}}dx \\
&= \frac{1}{\Gamma_\lambda(1)}\int_0^\infty (1+\lambda x)^{-\frac{1}{\lambda}(1-t)}dx = \frac{1}{\Gamma_\lambda(1)}\frac{1}{1-\lambda-t} \\
&= \frac{1}{\Gamma_\lambda(1)}\frac{1}{1-\lambda-t}(1+\lambda t)^{-\frac{1}{\lambda}}\cdot(1+\lambda t)^{\frac{1}{\lambda}} \\
&= \frac{1}{\Gamma_\lambda(1)}\left(\sum_{l=0}^\infty d_{l,\lambda}\frac{t^l}{l!}\right)\left(\sum_{m=0}^\infty (1)_{m,\lambda}\frac{t^m}{m!}\right) \\
&= \frac{1}{\Gamma_\lambda(1)}\sum_{n=0}^\infty\left(\sum_{l=0}^n\binom{n}{l}d_{l,\lambda}(1)_{n-l,\lambda}\right)\frac{t^n}{n!}.
\end{aligned} \tag{3.15}
$$

Therefore, by (3.14) and (3.15), we obtain the following theorem.

Theorem 3.1. *Let $S_{1,\lambda}(n,k)$ be the degenerate Stirling numbers of the first kind given in* (1.9) *or* (2.3), *and let $d_{l,\lambda}$ be the degenerate derangement numbers given in* (3.4). *Then we have*

$$
\sum_{k=n}^\infty \frac{S_{1,\lambda}(k,n)n!}{(1-\lambda)(1-2\lambda)\cdots(1-(k+1)\lambda)} = \sum_{l=0}^n\binom{n}{l}d_{l,\lambda}(1)_{n-l,\lambda},
$$

where $n \geq 0$, $k \in \mathbb{N}$, and $\lambda \in (0, \frac{1}{k+1})$.

From (3.4), we have

$$
\begin{aligned}
\sum_{n=0}^\infty d_{n,\lambda}\frac{t^n}{n!} &= \frac{1}{1-\lambda-t}(1+\lambda t)^{-\frac{1}{\lambda}} = \frac{1}{1-\lambda}\left(\frac{1}{1-\frac{t}{1-\lambda}}\right)(1+\lambda t)^{-\frac{1}{\lambda}} \\
&= \frac{1}{1-\lambda}\sum_{m=0}^\infty\left(\frac{t}{1-\lambda}\right)^m\sum_{l=0}^\infty(-1)^l <1>_{l,\lambda}\frac{t^l}{l!} \\
&= \frac{1}{1-\lambda}\sum_{n=0}^\infty\left(n!\sum_{l=0}^n\left(\frac{1}{1-\lambda}\right)^{n-l}(-1)^l<1>_{l,\lambda}\frac{1}{l!}\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^\infty\left(n!\sum_{l=0}^n\left(\frac{1}{1-\lambda}\right)^{n-l+1}\frac{(-1)^l}{l!}<1>_{l,\lambda}\right)\frac{t^n}{n!}.
\end{aligned} \tag{3.16}
$$

Comparing the coefficients on both sides of (3.16), we have

$$
d_{n,\lambda} = n!\sum_{l=0}^n\left(\frac{1}{1-\lambda}\right)^{n-l+1}\frac{(-1)^l}{l!}<1>_{l,\lambda}\ . \tag{3.17}
$$

Thus we get the following theorem.

Theorem 3.2. *Let $d_{l,\lambda}$ be the degenerate derangement numbers given in* (3.4). *Then, for $n \geq 0$, we have*

$$d_{n,\lambda} = n! \sum_{l=0}^{n} \left(\frac{1}{1-\lambda} \right)^{n-l+1} \frac{(-1)^l}{l!} < 1 >_{l,\lambda} .$$

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